How airplanes fly and ships sail

The discussion here is a supplement of calculations and pictures to Eremenko's note `https://www.math.purdue.edu/~eremenko/dvi/airplanes.pdf`, which has more information and references. See also sections 2.1.1, 3.4.1, and 4.2 of Fisher's *Complex Variables*.

Let $v(z)$ be a vector field (representing the wind) which is incompressible (i.e. divergence free, $\partial_x \operatorname{Re} v + \partial_y \operatorname{Im} v = 0$) and irrotational (i.e. curl free, $\partial_x \operatorname{Im} v - \partial_y \operatorname{Re} v = 0$).

This is the nicest kind of fluid flow, with no vortices, turbulence, viscosity, etc. Air can behave like this under favorable conditions. We are interested in flow around an impermeable object. We represent the object by D, a domain in the complex plane. Impermeability means that v is defined on the exterior of D and is tangent to the boundary of D. Thus there is no drag; neglecting drag makes sense for a sufficiently aerodynamic object, such as an airplane wing or a taut sail nearly parallel to the wind.

We begin with the case of a rotating cylinder, as analyzed in Figure 1.

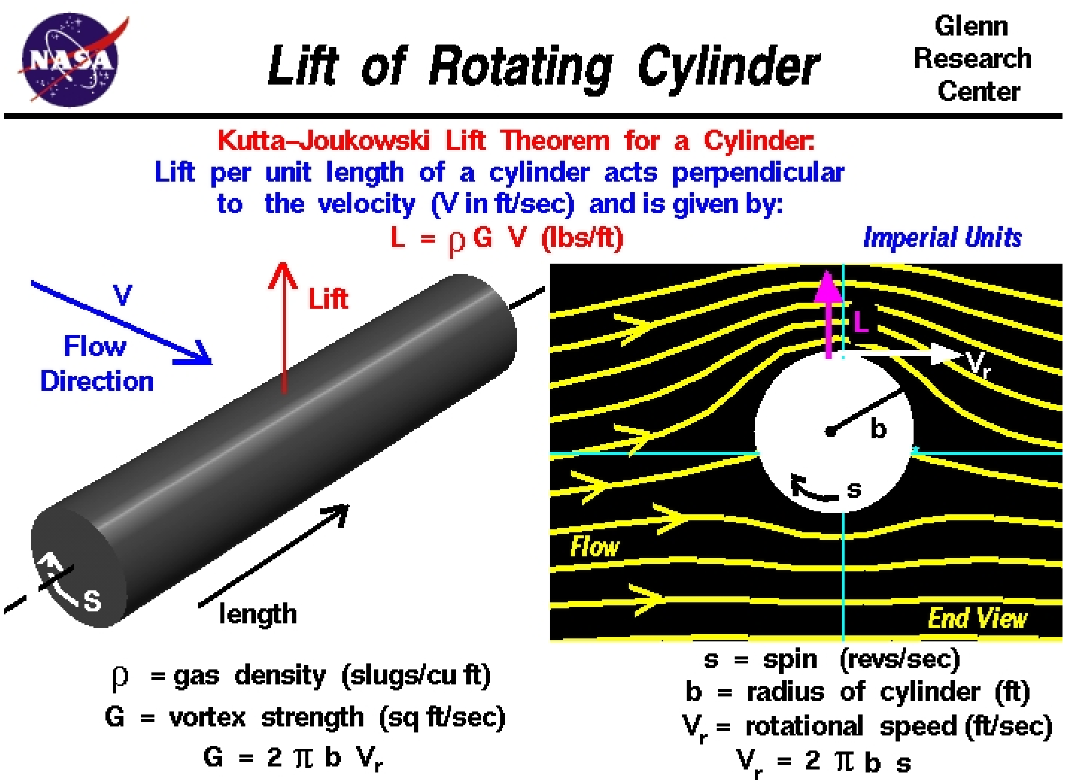


FIGURE 1. Let's try to understand some of this discussion from `https://www.grc.nasa.gov/www/k-12/airplane/cyl.html`.

Kiril Datchev, December 2, 2024. These notes are under development. Questions, comments, and corrections are gratefully received at kdatchev@purdue.edu.

A more basic version of the flow depicted in Figure 1, where the backgroud velocity is absent, is the circular vector field $v(z) = -ic/\bar{z}$ with c real, where D is the disk $\{z\colon |z| < R\}$ for some $R > 0$. This corresponds to a cylinder rotating in still air. See Figure 2.

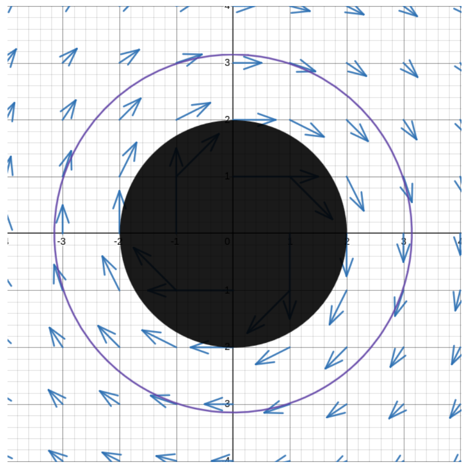


FIGURE 2. A circular flow around a circular object: $D = \{z\colon |z| < 2\}$, $v(z) = -ic/\bar{z}$, $f(z) = ic/z$, $F(z) = ic\log z$, with $c = 15$. The purple circle is the level set $\operatorname{Im} F(z) = 17.2$. You can adjust the parameters here: `https://www.desmos.com/calculator/sd2pssl2gx`.

To see how the background velocity interacts with this rotation, we use some complex analysis. When the vector field $v(z)$ is incompressible and irrotational, the function f defined by $f(z) = \overline{v(z)}$ is analytic because f obeys the Cauchy–Riemann equations. The function f is called the *complex velocity*.

Let F be a complex antiderivative of f, so $F'(z) = f(z)$. The function F is called a *complex potential*. It is significant because for every z, $v(z)$ is tangent to the level set of $\operatorname{Im} F$ at z. To prove this, notice that if $s'(t) = v(s(t))$ (so s is a parametrized curve representing the trajectory of a particle in the wind) then

$$\frac{d}{dt}F(s(t)) = F'(s(t))s'(t) = f(s(t))v(s(t)) = |f(s(t))|^2.$$

Taking the imaginary part of both sides gives $\frac{d}{dt}\operatorname{Im} F(s(t)) = 0$, so each s lives on a level set of $\operatorname{Im} F$ and that means that for every z, $v(z)$ is tangent to the level set of $\operatorname{Im} F$ at z.

Flow around a cylinder. Let D be a disk centered at the origin. Let $v_\infty > 0$ be the background velocity of the wind (i.e. the velocity far away from D), and c be a real number which will measure circulation around D. We will check that if

$$f(z) = v_\infty + \frac{ic}{z} - \frac{v_\infty R^2}{z^2}, \tag{1}$$

then $v(z) = \overline{f(z)}$ is tangent to the circle $|z| = R$. To check this, note that we are checking that $v(z)$ is perpendicular to z when $|z| = R$, i.e. as on page 8 of Fisher that $\operatorname{Re} z\overline{v(z)} = \operatorname{Re} zf(z) = 0$.

So we calculate

$$\operatorname{Re} z f(z) = \operatorname{Re} v_\infty z + ic - \frac{v_\infty R^2}{z} = v_\infty (\operatorname{Re} z) - v_\infty R^2 (\operatorname{Re} z)/|z|^2,$$

which is 0 when $|z| = R$. Another way to check the tangency requirement is to calculate

$$F(z) = v_\infty z + ic \log z + v_\infty R^2/z, \qquad \operatorname{Im} F(z) = v_\infty \operatorname{Im} z + ic \log |z| - v_\infty \operatorname{Im} z R^2/|z|^2,$$

and note that if $|z| = R$ then $\operatorname{Im} F(z) = c \log |R|$ which is independent of z. Thus the circle $|z| = R$ is contained in the level set $\operatorname{Im} F(z) = c \log |R|$ and since v is tangent to the level set it is also tangent to the circle. See Figure 3.

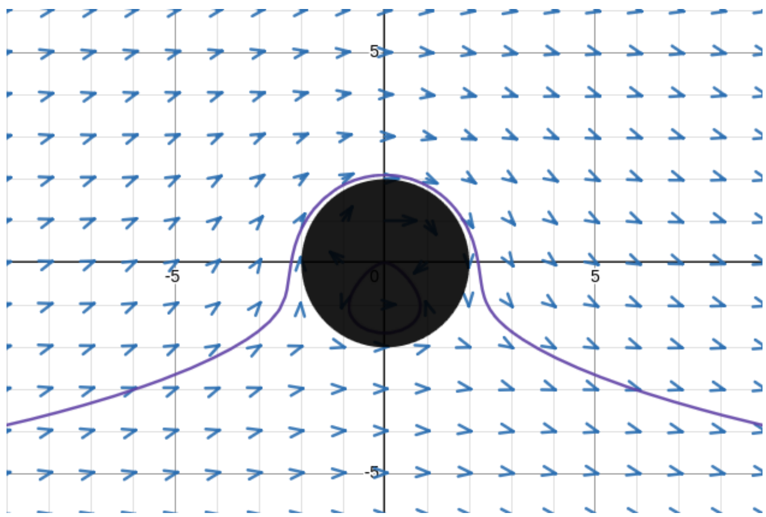


FIGURE 3. A plot when $v_\infty = 1$, $c = 2.5$, and $R = 2$. The purple curve is $\operatorname{Im} F(z) = 2$ and depicts a flowline going very close to the top of the wing. You can adjust the parameters here: `https://www.desmos.com/calculator/bxv6wo7upp`.

It turns out that the above are *all* the possibilities for f. More precisely, given constants $v_\infty > 0$ and $R > 0$, if 1) f is analytic on $\{z\colon |z| > R\}$, 2) f obeys $\lim_{|z|\to\infty} f(z) = v_\infty$, and 3) $v(z) = \overline{f(z)}$ is tangent to $|z| = R$ (i.e. we have $\operatorname{Re} z f(z) = 0$ when $|z| = R$) then there is a real c such that f is given by (1). This is proved in the section 'Uniqueness' below.

Lift force. The lift force on the wing is

$$L = \overline{\frac{i\rho}{2} \int_{\partial D} f(z)^2 dz} = 2\pi i \rho v_\infty c, \tag{2}$$

where ρ is the density of the fluid. In the notation of Figure 1, the lift is $\rho G V$, where their V is our v_∞, and $G = 2\pi V_r$, where their V_r is our c.

This formula is called Blasius' Theorem, and we derive it via Bernoulli's principle, following the first few pages of Eremenko's notes. The first part of the calculation is easier if we write in terms of ordered pairs of real numbers rather than in terms of complex numbers. Write the vector field as $v = (v_1, v_2)$, so that a particle following it obeys

$$x'(t) = v_1(x(t), y(t)), \qquad y'(t) = v_2(x(t), y(t)). \tag{3}$$

Differentiating gives

$$x''(t) = \partial_x v_1 x'(t) + \partial_y v_1 y'(t), \qquad y''(t) = \partial_x v_2 x'(t) + \partial_y v_2 y'(t),$$

where we are abbreviating $\partial_x v_1(x(t), y(t))$ as $\partial_x v_1$, and similarly for $\partial_y v_1$, etc. Plugging in (3), and continuing the pattern of abbreviation, gives

$$x''(t) = \partial_x v_1 v_1 + \partial_y v_1 v_2, \qquad y''(t) = \partial_x v_2 v_1 + \partial_y v_2 v_2.$$

Plugging in the irrotationality equation $\partial_y v_1 = \partial_x v_2$ gives

$$x''(t) = \partial_x v_1 v_1 + \partial_x v_2 v_2, \qquad y''(t) = \partial_y v_1 v_1 + \partial_y v_2 v_2.$$

or

$$x''(t) = \frac{1}{2}\partial_x |v|^2, \qquad y''(t) = \frac{1}{2}\partial_y |v|^2,$$

or

$$(x''(t), y''(t)) = \frac{1}{2}\nabla |v(x(t), y(t))|^2,$$

where we have now stopped abbreviating. We invoke the physical principles that force is proportional to acceleration and that force is the negative gradient of pressure to conclude that

$$\frac{\rho}{2}\nabla |v(x, y)|^2 = -\nabla p(x, y),$$

where ρ is the density of the fluid (assumed constant) and $p(x, y)$ is the pressure at (x, y). If two functions have the same gradient then the difference is a constant, so we conclude that

$$\frac{\rho}{2}|v(x, y)|^2 + p(x, y) = \text{constant} \tag{4}$$

Equation (4) is called *Bernoulli's principle*. The quantity $\frac{\rho}{2}|v|^2$ is a kind of kinetic energy, while the pressure p is a kind of potential energy, so Bernoulli's principle is a version of the statement that the sum of kinetic and potential energy is a constant. Note also that a greater speed corresponds to a lesser pressure, and vice versa.

Let us now compute the force on the cross section D as a complex line integral. At each point, the force is given by the pressure, and is directed inward (trying to push into the impremeable object). Thus, if $\gamma\colon [t_0, t_1] \to \partial D$ parametrizes boundary so that obstacle is to the left, then lifting force is

$$L = \int_{t_0}^{t_1} ip(\gamma(t))\gamma'(t)dt = -i\frac{\rho}{2}\int_{t_0}^{t_1} |v(\gamma(t))|^2\gamma'(t)dt,$$

where for the second equals sign we used (4) and the fact that $\gamma(t_0) = \gamma(t_1)$. But the velocity must be proportional to γ' at each point, since the object is impermeable, so there is a real function $\lambda(t)$ such that write $v(\gamma(t)) = \lambda(t)\gamma'(t)$. Plugging this in, and using the fact that $\bar{v} = f$, we obtain

$$|v|^2\gamma' = \lambda^2\gamma'\overline{\gamma'}\gamma' = v^2\overline{\gamma'} = \overline{f^2\gamma'},$$

which gives

$$L = -i\frac{\rho}{2}\int_{t_0}^{t_1} \overline{f(\gamma(t))^2\gamma'(t)}dt = \overline{\frac{i\rho}{2}\int_{\partial D} f(z)^2 dz},$$

as desired for the first equals of (2).

To get the second equals of (2), we expand f in a Laurent series as

$$f(z) = v_\infty + \frac{ic}{z} + \cdots ,$$

which implies

$$f(z)^2 = v_\infty^2 + \frac{2icv_\infty}{z} + \cdots ,$$

so that the residue of $f(z)^2$ is $2icv_\infty$ and

$$\int_{\partial D} f(z)^2 dz = 2\pi i(2icv_\infty) = -4\pi c v_\infty,$$

which gives the second equals of (2).

More general cross sections. We can treat many other cross sections D by using a mapping, or change of variables, or change of coordinates, to reduce to the case of a circle. The most general result of this kind is the Riemann mapping theorem.[1] See the rest of Chapter 3 of Fisher for more discussion and various examples and general methods. We will just look at a few examples.

The case of a segment. Let D be given by the segment from $Le^{-i\alpha}/2$ to $-Le^{i\alpha}/2$ for some real L and α. We start with the case $L = 4$, $\alpha = 0$, for which we use the mapping $z \mapsto w(z)$ defined by

$$z = w(z) + \frac{1}{w(z)}.$$

This is called the *Joukowski* mapping. To see that the exterior regions $\{z\colon \operatorname{Im} z \neq 0 \text{ or } |\operatorname{Re} z| > 2\}$ and $\{w\colon |w| > 1\}$ are in one-to-one correspondence, note that

$$\operatorname{Re} z = \operatorname{Re} w(1 + |w|^{-2}), \qquad \operatorname{Im} z = \operatorname{Im} w(1 - |w|^{-2}).$$

and so if $R > 1$ then the circle $|w| = R$ is mapped to the ellipse passing through the points $\pm(R + R^{-1})$ and $\pm i(R - R^{-1})$. See Figure 4. We can solve for $w(z)$ using the quadratic formula to obtain

$$w(z) = \frac{z}{2}\Big(1 + \sqrt{1 - \frac{4}{z^2}}\Big). \tag{5}$$

We can use as a complex potential the function

$$F(z) = v_\infty w(z) + ic \log w(z) + v_\infty / w(z) = v_\infty z + ic \log z + \cdots , \tag{6}$$

where the first equals sign is the definition of F, and for the second we used $w(z) - z \to 0$ as $z \to \infty$; the $\cdots$ in (6) is a bounded analytic function, defined in the complement of D, with a series $\sum_{n=0}^\infty a_n z^{-n}$ that we do not need to compute.

[1] The Riemann mapping theorem applies as long as D is a connected, simply connected, bounded, closed set. To treat the boundary of D we must assume the boundary is a simple closed continuous curve and apply Carathéodory's theorem; see for example Theorem 3.1 of *Harmonic Measure* by Garnett and Marshall.

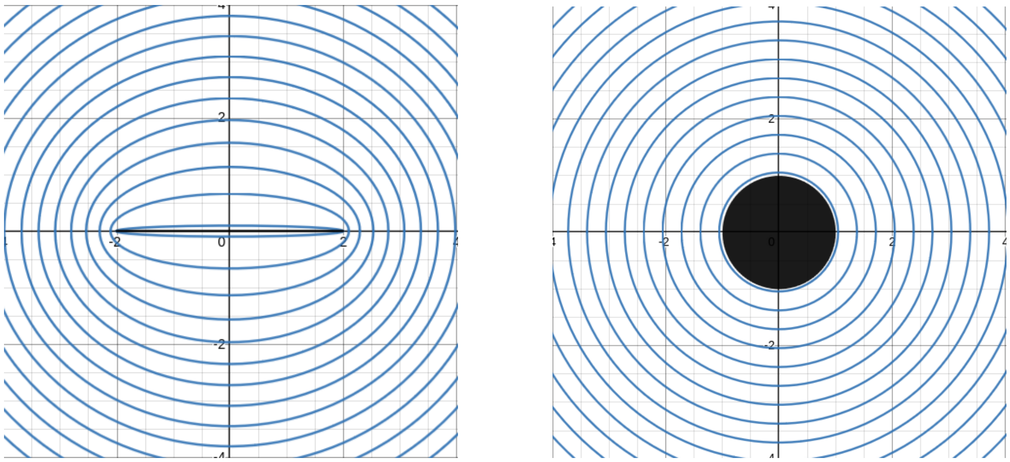


FIGURE 4. The correspondence $z = w + \frac{1}{w}$. The circles $|w| = R$ for various values of $R > 1$ on the right, with the respective ellipses on the left. Thus the exterior regions $\{z\colon \operatorname{Im} z \neq 0 \text{ or } |\operatorname{Re} z| > 2\}$ and $\{w\colon |w| > 1\}$ are in one-to-one correspondence. See `https://www.desmos.com/calculator/kqeypion7o` and `https://www.desmos.com/calculator/1hdmplt0xa`.

For the more general segment from $Le^{-i\alpha}/2$ to $-Le^{i\alpha}/2$, we multiply z by $4e^{i\alpha}/L$ to map back to the segment from -2 to 2, and give w the same factor so that we maintain $w(z) - z \to 0$ as $z \to \infty$. That gives

$$w(z) = \frac{z}{2}\Big(1 + \sqrt{1 - \frac{L^2}{4e^{2ia}z^2}}\Big). \tag{7}$$

We once again get F of the form (6), but now with w given by (7).

To determine c, we use *Kutta's principle*, which says that if there is a trailing sharp edge, then $f(z) = 0$ there so that the flow leaves the edge smoothly. In the case above, the trailing sharp edge is at $z = Le^{-i\alpha}/2$, i.e. $w = Le^{-i\alpha}/4$. To compute c, we substitute $w = Le^{-i\alpha}/4$ and $R = |w| = L/4$ into $v_\infty + \frac{ic}{w} + \frac{R^2 v_\infty}{w^2} = 0$ and solve for c to get

$$c = v_\infty L \sin\alpha/2.$$

Thus the magnitude of the lift is

$$\pi\rho v_\infty^2 L \sin\alpha.$$

Let's say for instance we have $\rho = 1\,kg/m^2$ (the density of air) $v_\infty = 10\,m/s$ (a nice breeze of about 22 miles per hour) $L = 1\,m$, $\sin\alpha = 1/5$ (our angle with the wind is about $11.5°$). That gives 20π Newtons of force or about the weight of a 2π kg mass per square meter of sail. If the sail is 10 square meters that gives about 20π kg, the size of a person.

Note that we are neglecting drag, so α must be small for this to be realistic.

The case of a Joukowski airfoil. We obtain a *Joukowski airfoil*, which is a classic airplane wing cross section, by using again $z = w + \frac{1}{w}$ but replacing the circle $|w| = 1$ with a circle $|w - p| = |1 - p|$, with p close to the origin: see Figure 6.

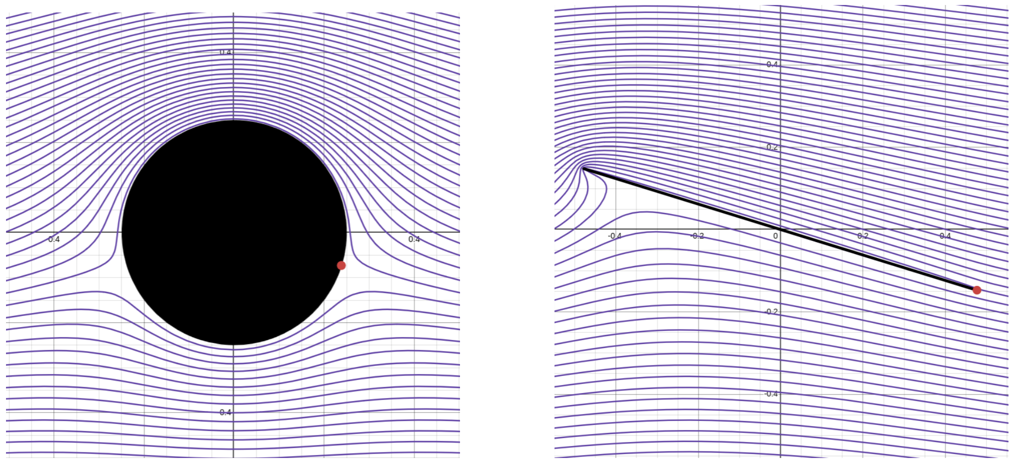


FIGURE 5. On the left, the flow with complex potential $w + ic\log(w) + \frac{1}{16w}$ around the circle $|w| = 1/4$, with $c = \sin\alpha/2$, $\alpha = 0.3$. On the left, the corresponding flow around the segment from $e^{-i\alpha}/2$ to $-e^{-i\alpha}/2$, under the correspondence $w(z) = \frac{z}{2}\left(1 + \sqrt{1 - \frac{1}{4e^{2i\alpha}z^2}}\right)$. The two red points are mapped to one another and represent the trailing edge of the wing; c is chosen so to make the flow smooth there. See `https://www.desmos.com/calculator/qmx8s0q9t9` and `https://www.desmos.com/calculator/dwiiowc77o`.

Since $w(2) = 1$, the trailing sharp edge at $z = 2$ is mapped to the point 1 on the circle. If we parametrize the circle with $w = p + |1-p|e^{it}$, then $w = 1$ corresponds to $e^{it} = (1-p)/|1-p|$, so $t = \arg(1-p)$. This corresponds to the case of a tilted segment with $\alpha = -t$, $\sin\alpha = \operatorname{Im} p/|1-p|$, and $L = 2R = 2|1-p|$. That yields

$$c = 2v_\infty \operatorname{Im} p,$$

and thus the magnitude of the lift force is

$$4\pi\rho v_\infty^2 \operatorname{Im} p,$$

for an airfoil corresponding to a segment of length 4, and for length L we accordingly multiply by $L/4$.

To plot this, use

$$v_\infty(\operatorname{Im}(w-p)) + c\log|w-p| - v_\infty R^2 \operatorname{Im}(w-p)/|w-p|^2 = C,$$

and solve for $\operatorname{Im} w$ to get

$$\operatorname{Im} w = \operatorname{Im} p + \frac{C - c\log|w-p|}{v_\infty(1 - R^2|w-p|^{-2})}$$

Uniqueness. Let's check that, given constants v_∞ and $R > 0$, with equation (1) we have found *all* the functions f such that

i) f is analytic on $\{z\colon |z| > R\}$ and continuous on $\{z\colon |z| \geq R\}$,
ii) $\lim_{|z|\to\infty} f(z) = v_\infty$, and
iii) $\operatorname{Re} z f(z) = 0$ when $|z| = R$.

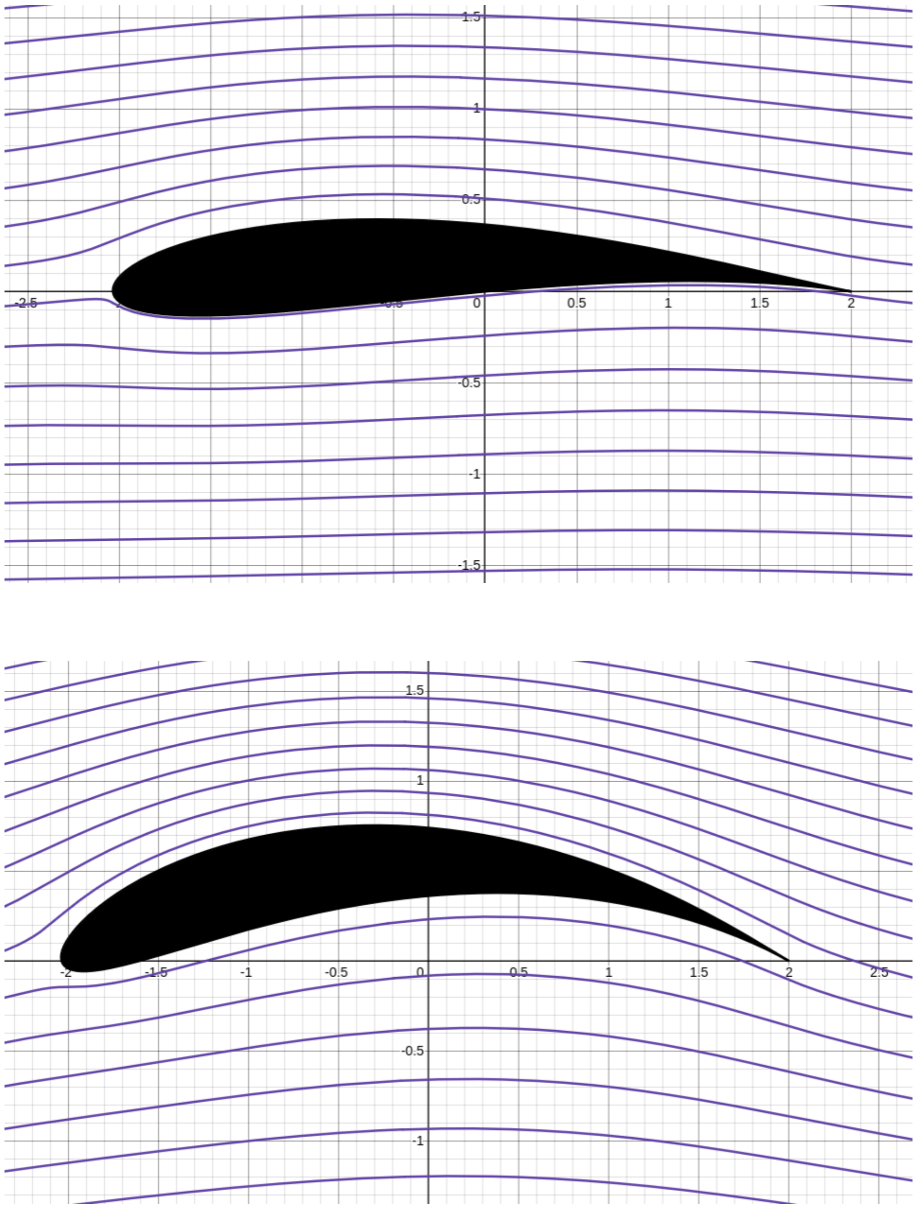


FIGURE 6. Flow around two Joukowski airfoils. The first is with $p = -.1 + .1i$ and the second with $p = -.1 + .3i$. The second generates three times the lift because the imaginary part of p is tripled. See `https://www.desmos.com/calculator/oowkgnuxkw`, and also `https://en.wikipedia.org/wiki/Airfoil`, especially the first image and video there.

This shows that the discussion of 'The flow around a cylinder' above covers *all* examples with $D = \{z\colon |z| < R\}$.

Start with an arbitrary such f. Then on $\{z\colon |z|>R\}$, f has a Laurent series expansion

$$f(z)=v_\infty+\sum_{n=1}^{\infty}a_n z^{-n}.$$

We now consider the difference

$$g(z)=f(z)-\Big(v_\infty+\frac{ic}{z}-\frac{v_\infty R^2}{z^2}\Big),$$

with $c=\operatorname{Im}a_1$. We will use a carefully chosen transformation to convert g into a function h such that

- iv) h is analytic on $\{z\colon |z|<1/R\}$ and continuous on $\{z\colon |z|\le 1/R\}$, and
- v) $\operatorname{Re}h(z)=0$ when $|z|=1/R$.
- vi) $h(0)$ is real.

Then by the maximum principle (see the section below) it will follow that $\operatorname{Re}h(z)=0$ on $\{z\colon |z|\le 1/R\}$. By the Cauchy–Riemann equations, $\operatorname{Im}h(z)=0$ on $\{z\colon |z|\le 1/R\}$ as well, and hence $g(z)=0$ on $\{z\colon |z|\ge R\}$.

To obtain h, note that

$$g(z)=\sum_{n=1}^{\infty}b_n z^{-n},$$

with b_1 real. Moreover, by direct calculation and using $\operatorname{Re}zf(z)=0$, we have $\operatorname{Re}zg(z)=0$ when $|z|=R$. Now let

$$h(z)=g(1/z)/z=\sum_{n=1}^{\infty}b_n z^{n-1}.$$

This h has properties iv) and v), so $h(z)=0$ on $\{z\colon |z|\le 1/R\}$ and $g(z)=0$ on $\{z\colon |z|\ge R\}$.

Maximum principle. The maximum principle states that if h is analytic on $\{z\colon |z|<1/R\}$ and continuous on $\{z\colon |z|\le 1/R\}$, then $\operatorname{Re}h$ attains its maximum and minimum values on $\{z\colon |z|=1/R\}$.

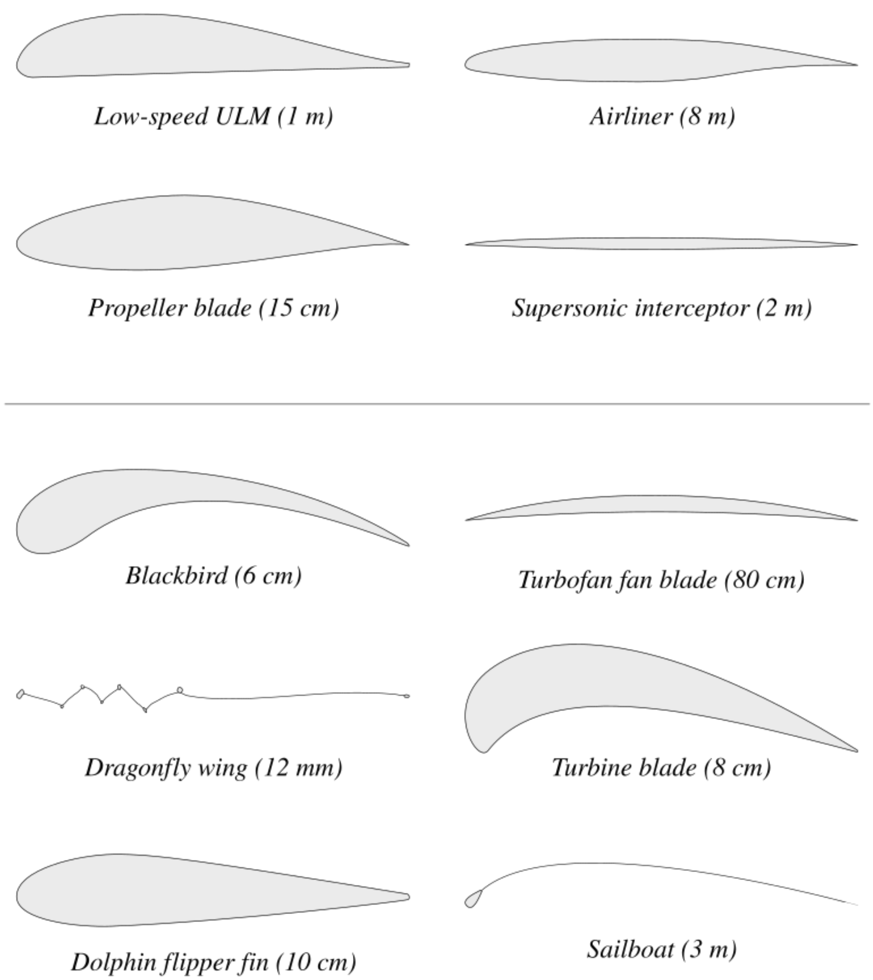
This follows from the fact that, by the Cauchy integral formula,

$$h(p)=\frac{1}{2\pi i}\int_{|z-p|=r}\frac{h(z)dz}{z-p}=\frac{1}{2\pi}\int_0^{2\pi}h(p+re^{i\theta})d\theta,\tag{8}$$

when $|p|<1/R$ and $r<1/R-|p|$; equation (8) is called the *mean value property* because the value of h at p is the average of the values of h on any circle centered at p, as long as the circle fits within the zone of analyticity $|z|<1/R$.

Let's show that the mean value property (8) implies the maximum principle. To do this, we show that if $\operatorname{Re}h$ attains a maximum at some p in $\{z\colon |z|<1/R\}$ then $\operatorname{Re}h$ is constant, and hence it attains the same maximum on $\{z\colon |z|=1/R\}$ as well. Note that $\operatorname{Re}h(p+re^{i\theta})=\operatorname{Re}h(p)$ for all $r<1/R-|p|$ and all θ because we have $\operatorname{Re}h(p+re^{i\theta})\le\operatorname{Re}h(p)$ by the fact that $\operatorname{Re}h(p)$ is maximal and if we ever had $\operatorname{Re}h(p+re^{i\theta})<\operatorname{Re}h(p)$ we would have $\frac{1}{2\pi}\int_0^{2\pi}\operatorname{Re}h(p+re^{i\theta})d\theta<\operatorname{Re}h(p)$, violating (8). This shows that $\operatorname{Re}h$ is constant on $\{z\colon |z-p|<1/R-|p|$. If we had $p=0$ then this reduces to $\operatorname{Re}h$ is constant on $\{z\colon |z|<1/R\}$. If $p\ne 0$, then repeat the argument with p

replaced by p', where p' is closer to 0 than p. Repeating the argument enough times proves that $\operatorname{Re} h$ is constant on $\{z\colon |z| < 1/R\}$. Applying the same result with $-h$ in place of h proves that $\operatorname{Re} h(p) \geq 0$ for all p.



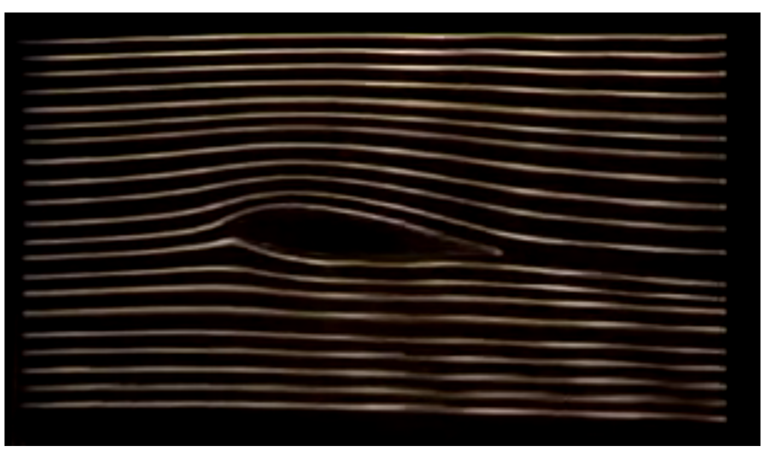


FIGURE 7. Above, examples of airfoils by Olivier Cleynen from `https://commons.wikimedia.org/w/index.php?curid=17568805`. Below, a still from a video by Holger Babinsky of smoke flowing around an airfoil in a wind tunnel from (`https://commons.wikimedia.org/wiki/File:Flow_over_aerofoils.webm`..